DERIVED EQUIVALENCE INDUCED BY n-TILTING MODULES

SILVANA BAZZONI, FRANCESCA MANTESE, AND ALBERTO TONOLO

Abstract. Let T_R be a right n-tilting module over an arbitrary associative ring R. In this paper we prove that there exists a n-tilting module T'_R equivalent to T_R which induces a derived equivalence between the unbounded derived category $\mathcal{D}(R)$ and a triangulated subcategory $\mathcal{E}_\perp$ of $\mathcal{D}(\mathrm{End}(T'))$ equivalent to the quotient category of $\mathcal{D}(\mathrm{End}(T'))$ modulo the kernel of the total left derived functor $-\otimes^{\mathbb{L}}_{S'} T'$. In case T_R is a classical n-tilting module, we get again the Cline-Parshall-Scott and Happel's results.

Introduction

Tilting theory generalizes the classical Morita theory of equivalences between module categories. Originated in the works of Gel'fand and Ponomariev, Brenner and Butler, Happel and Ringel [4, 7, 17], it has been generalized in various directions. In the recent literature, given an associative ring R with $0 \neq 1$, a right R-module T_R is said to be n-tilting if the following conditions are satisfied:

(T1) there exists a projective resolution of right R-modules

$$0 \to P_n \to ... \to P_1 \to P_0 \to T \to 0;$$

(T2) $\mathrm{Ext}^i_R(T, T^{(\alpha)}) = 0$ for each $i > 0$ and each cardinal α;

(T3) there exists a coresolution of right R-modules

$$0 \to R \to T_0 \to T_1 \to ... \to T_m \to 0,$$

where the T_i's are direct summands of arbitrary direct sums of copies of T.

If the projectives P_i's in (T1) can be assumed finitely generated, then the n-tilting module T_R is said *classical n-tilting*.

Let us denote by $S = \mathrm{End}(T_R)$ the endomorphism ring of T and by $KE_i(T)$ and $KT_i(T)$, $0 \leq i \leq n$, the following classes

$$KE_i(T) = \{M \in \text{Mod-}R : \mathrm{Ext}^j_R(T, M) = 0 \text{ for each } 0 \leq j \neq i\},$$

$$KT_i(T) = \{N \in \text{Mod-}S : \mathrm{Tor}^S_j(N, T) = 0 \text{ for each } 0 \leq j \neq i\}.$$

In 1986 Miyashita [21] proved that if T_R is a classical n-tilting, then the functors $\mathrm{Ext}^i_R(T, -)$ and $\mathrm{Tor}^S_i(-, T)$ induce equivalences between the classes $KE_i(T)$ and $KT_i(T)$.

In the same years, works of several authors showed that the natural context for studying equivalences induced by classical tilting modules is that of derived categories. In particular Cline, Parshall and Scott [8], generalizing a result of Happel [16], proved that a classical n-tilting module T_R provides a derived equivalence between the bounded derived categories $\mathcal{D}^b(R)$ and $\mathcal{D}^b(S)$ of bounded cochain complexes of right R- and S- modules.

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In 1988 Facchini [10, 11] proved that, over a commutative domain, the divisible module ∂ introduced by Fuchs [12] is an infinitely generated 1-tilting module and it provides a pair of equivalences

$$KE_0(\partial) \underset{-\otimes\partial}{\overset{\mathrm{Hom}(\partial,-)}{\rightleftarrows}} KT_0(\partial) \cap I\text{-}Cot, \quad KE_1(\partial) \underset{Tor_1(-,\partial}{\overset{\mathrm{Ext}^1(\partial,-)}{\rightleftarrows}} KT_1(\partial) \cap I\text{-}Cot$$

between the category $KE_0(\partial)$ of all divisible modules and the category $KT_0(\partial) \cap I\text{-}Cot$ of all *I-reduced I-cotorsion modules*, and the category $KE_1(\partial)$ of all reduced modules and the category $KT_1(\partial) \cap I\text{-}Cot$ of all *I-divisible I-cotorsion* modules, respectively. In 1995 Colpi and Trlifaj [9] started the study in general of 1-tilting modules. They realized that it can be useful to "change slightly" the tilting module to realize a good equivalence theory. They proved that if T_R is a 1-tilting module, there exists another 1-tilting module T'_R *equivalent to* T_R (i.e. $KE_0(T) = KE_0(T')$), with endomorphism ring $S' = \mathrm{End}(T')$, such that the functors $\mathrm{Hom}_R(T', -)$ and $- \otimes_{S'} T'$ induce an equivalence between $KE_0(T) = KE_0(T')$ and its image class in Mod-S'. Moreover T' results to be a finitely presented S'-module. In 2001 Gregorio and Tonolo extended this result proving the existence of a pair of equivalences

$$KE_i(T') \underset{\mathrm{Tor}_i^{S'}(-,T')}{\overset{\mathrm{Ext}^i_R(T',-)}{\rightleftarrows}} KT_i(T') \cap Cost(T'), \quad i = 1, 2$$

where $Cost(T')$ is the class of *costatic* right S'-modules (see [15]).

In 2009 Bazzoni [3] gives a better understanding of the whole situation in the setting of derived categories proving that for a 1-tilting module T_R it is possible to find an equivalent 1-tilting module T' which induces a derived equivalence between the unbounded derived category $\mathcal{D}(R)$ and the quotient category of $\mathcal{D}(S')$ modulo the full triangulated subcategory $\mathrm{Ker}(- \otimes^{\mathbb{L}}_{S'} T')$, namely the kernel of the total left derived functor of the functor $- \otimes_{S'} T'$.

In this paper we generalize the Bazzoni's result to a general n-tilting module T_R. We prove the existence of a *good* n-tilting module T'_R *equivalent* to T_R (see Definition 1.1) which, also in such a case, provides a derived equivalence between the unbounded derived category $\mathcal{D}(R)$ and a triangulated subcategory $\mathcal{E}_\perp$ of $\mathcal{D}(\mathrm{End}(T'))$. The category $\mathcal{E}_\perp$ results to be equivalent to the quotient category of $\mathcal{D}(\mathrm{End}(T'))$ modulo the kernel of the total left derived functor $- \otimes^{\mathbb{L}}_{S'} T'$. Moreover, as done in [20] in the contravariant case, we interpret the derived equivalence at the level of stalk complexes obtaining on the underlying module categories a generalization of the Miyashita equivalences.

1. n-tilting classes

In 2004 Bazzoni (see [2]) proved that T_R is a n-tilting module if and only if the classes

$$T^{\perp_\infty} := \{M_R : \mathrm{Ext}^i_R(T, M) = 0 \text{ for each } i > 0\}$$

and

$$\mathrm{Gen}_n(T) := \{M_R : \exists\, T^{(\alpha_n)} \to ... \to T^{(\alpha_1)} \to M \to 0, \text{ for some cardinals } \alpha_i\}$$

coincide.

Definition 1.1. Two n-tilting right R-modules T_R and T'_R are said *equivalent* if $\mathrm{Gen}_n(T_R) = \mathrm{Gen}_n(T'_R)$.

An arbitrary direct sum of copies of a n-tilting module is a n-tilting module equivalent to the original one. Therefore equivalent tilting modules can have completely different endomorphism rings.

Definition 1.2. We say that T_R is a *good* n-tilting module if it is n-tilting and it satisfies the condition

(T3') there is an exact sequence

$$0 \to R \to T_0 \to T_1 \to ... \to T_n \to 0$$

where the T_i's are direct summands of finite direct sums of copies of T.

Each classical n-tilting module is good [14, Section 5.1].

Proposition 1.3. *For any n-tilting module T_R there exists an equivalent good n-tilting module T'_R such that*

$$KE_i(T) = KE_i(T') \text{ for each } i \geq 0.$$

Proof. Let T_R be a n-tilting module. If it is classical, then T already satisfies (T3'). Otherwise, from condition $(T3)$ we easily get the exact sequence

$$0 \to R \to T_0 \to ... \to T_{n-2} \to T_{n-1} \oplus T_n^{(\omega)} \to T_n \oplus T_n^{(\omega)} \to 0$$

that can be rewritten in the form

$$0 \to R \to T_0 \to ... \to T_{n-2} \to T_{n-1} \oplus T_n^{(\omega)} \to T_n^{(\omega)} \to 0.$$

With the same argument we get the exact sequence

$$0 \to R \to ... \to T_{n-3} \to T_{n-2} \oplus (T_{n-1} \oplus T_n^{(\omega)})^{(\omega)} \to T_{n-1} \oplus T_n^{(\omega)} \oplus (T_{n-1} \oplus T_n^{(\omega)})^{(\omega)} \to T_n^{(\omega)} \to 0,$$

and hence the exact sequence

$$0 \to R \to T_0 \to ... \to T_{n-3} \to T_{n-2} \oplus T_{n-1}^{(\omega)} \oplus T_n^{(\omega)} \to T_{n-1}^{(\omega)} \oplus T_n^{(\omega)} \to T_n^{(\omega)} \to 0.$$

Iterating this procedure we get an exact sequence

$$0 \to R \to T_0 \oplus T_1^{(\omega)} \oplus ... \oplus T_n^{(\omega)} \to ... \to T_{n-2}^{(\omega)} \oplus T_{n-1}^{(\omega)} \oplus T_n^{(\omega)} \to T_{n-1}^{(\omega)} \oplus T_n^{(\omega)} \to T_n^{(\omega)} \to 0.$$

Let $T' = T_0 \oplus T_1^{(\omega)} \oplus ... \oplus T_n^{(\omega)}$; since T' is a direct summand of a direct sum of copies of T, we have

$$\mathrm{Gen}_n(T') \subseteq \mathrm{Gen}_n(T) = T^{\perp_\infty} \subseteq T'^{\perp_\infty},$$

and T' satisfies properties $(T1)$ and $(T2)$ of tilting modules. Since by construction it satisfies also property (T3'), we have $\mathrm{Gen}_n(T') = T'^{\perp_\infty}$ and T' is the wanted good n-tilting equivalent to T.

Finally, since $\mathrm{Ker}\,\mathrm{Ext}^j(T, -) = \mathrm{Ker}\,\mathrm{Ext}^j(T_0 \oplus ... \oplus T_n, -) = \mathrm{Ker}\,\mathrm{Ext}^j(T', -)$, we conclude that $KE_i(T) = KE_i(T')$ for each $i \geq 0$. $\square$

A good n-tilting module has an endomorphism ring S sufficiently large to permit to build a good equivalence theory between the unbounded derived categories $\mathcal{D}(R)$ and $\mathcal{D}(S)$. In the sequel we will work directly with good n-tilting modules.

Proposition 1.4. *Let T_R be a good n-tilting module and $S = \mathrm{End}(T_R)$. Then ${}_S T$ has a projective resolution*

$$0 \to Q_n \to ... \to Q_0 \to {}_S T \to 0$$

where the Q_i's are direct summand of a finite direct sum of copies of S, $\mathrm{Ext}^i_S(T, T) = 0$ for each $i \geq 0$, and $R \cong \mathrm{End}({}_S T)$.

Proof. By Definition 1.2 there is an exact sequence

$$0 \to R \to T_0 \to T_1 \to ... \to T_n \to 0$$

with the T_i's direct summands of T^m for a suitable $m \in \mathbb{N}$. Denote by K_i the kernel of the map $T_i \to T_{i+1}$, $1 \le i \le n-1$. Applying the contravariant functor $\mathrm{Hom}_R(-, T)$ we get easily by dimension shifting that

$$0 = \mathrm{Ext}^i_R(K_j, T) \text{ for each } 1 \le j \le n-1, \text{ and } i \ge 1.$$

Therefore we have the exact sequence

$$(\dagger) \quad 0 \to \mathrm{Hom}_R(T_n, T) \to \mathrm{Hom}_R(T_{n-1}, T) \to ... \to \mathrm{Hom}_R(T_1, T) \to \mathrm{Hom}_R(T_0, T) \to {}_S T \to 0$$

where each $\mathrm{Hom}_R(T_i, T)$ is a direct summand of $\mathrm{Hom}_R(T^m, T) = S^m$ and hence a finitely generated projective S-module. Given a right R-module M, let us denote for semplicity by M^* the left S-module $\mathrm{Hom}_R(M, T)$, by M^{**} the right R-module $\mathrm{Hom}_S(M^*, T)$, and by δ_M the evaluation map $M \to M^{**}$. The modules K_i^* are the cokernels of the morphisms $\mathrm{Hom}_R(T_{i+1}, T) \to \mathrm{Hom}_R(T_i, T)$, $1 \le i \le n-1$. Applying to $(\dagger)$ the contravariant functor $\mathrm{Hom}_S(-, T)$ we get the following commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(T,T) = R^{**} \arrow[r] & T_0^{**} \arrow[r] & K_1^{**} \arrow[r] & \mathrm{Ext}^1_S(T,T) \arrow[r] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[u, "\delta_R"] & T_0 \arrow[r] \arrow[u, "\delta_{T_0}"] & K_1 \arrow[r] \arrow[u, "\delta_{K_1}"] & 0 &
\end{tikzcd}$$

$$\cdots$$

$$\begin{tikzcd}
0 \arrow[r] & K_{n-1}^{**} \arrow[r] & T_{n-1}^{**} \arrow[r] & T_n^{**} \arrow[r] & \mathrm{Ext}^1_S(K_{n-1}^*, T) \arrow[r] & 0 \\
0 \arrow[r] & K_{n-1} \arrow[r] \arrow[u, "\delta_{K_{n-1}}"] & T_{n-1} \arrow[r] \arrow[u, "\delta_{T_{n-1}}"] & T_n \arrow[r] \arrow[u, "\delta_{T_n}"] & 0 &
\end{tikzcd}$$

Since the δ_{T_i}'s are isomorphisms we get

$$\mathrm{Ext}^1_S(T,T) = 0 \text{ and } 0 = \mathrm{Ext}^1_S(K_i^*, T) \cong \mathrm{Ext}^{i+1}_S(T,T) \text{ for each } 1 \le i \le n-1,$$

and $R \cong \mathrm{Hom}_S(T, T)$.

□

Lemma 1.5 (Lemmas 1.8, 1.9 [21])**.** *Let T_R be a good n-tilting and $S = \mathrm{End}\, T$. For any right R-module M in $T^{\perp_\infty}$ and any right projective S-module P_S, we have*

(1) $\mathrm{Tor}^S_i(\mathrm{Hom}_R(T, M), T) = 0$ *for each* $i > 0$.
(2) $\mathrm{Hom}_R(T, M) \otimes_S T \cong M, \quad f \otimes t \mapsto f(t)$
(3) $\mathrm{Ext}^i_R(T, P \otimes_S T) = 0$ *for each* $i > 0$.

If T_R is a classical n-tilting module, then

(4) $P \cong \mathrm{Hom}_R(T, P \otimes_S T), \quad p \mapsto (f : t \mapsto p \otimes t)$.

Proof. Everything except condition (3) follows by the quoted lemmas in [21]. If $P \le^{\oplus} S^{(\alpha)}$ we have

$$\mathrm{Ext}^i_R(T, P \otimes_S T) \le^{\oplus} \mathrm{Ext}^i_R(T, S^{(\alpha)} \otimes_S T) = \mathrm{Ext}^i_R(T, T^{(\alpha)}) = 0.$$

□

2. Tilting equivalences in derived categories

In the sequel, for any ring R, we denote by $\mathcal{K}(R)$ the homotopy category of unbounded complexes of right R-modules and by $\mathcal{D}(R)$ the associated derived category. Given an object $M \in \text{Mod-}R$, we continue to denote by M also the *stalk complex* in $\mathcal{D}(R)$ associated to M, i.e. the complex with M concentrated in degree zero. Any complex $C^\bullet \in \mathcal{D}(R)$ admits a K-injective resolution, i.e. a complex $\underline{\mathbf{i}}C^\bullet$ quasi-isomorphic to $C^\bullet$ whose terms are injective modules. Similarly, any complex $C^\bullet \in \mathcal{D}(R)$ admits a K-projective resolution, i.e. a complex $\underline{\mathbf{p}}C^\bullet$ quasi-isomorphic to $C^\bullet$ whose terms are projective modules (see for instance [5]). This result guarantees the existence of the total derived functor of any additive functor defined on module categories.

Given any covariant left exact functor $H : \text{Mod-}R \to \text{Mod-}S$, we denote by $\mathbb{R}\,H$ its total right derived functor defined on $\mathcal{D}(R)$. For any $C^\bullet \in \mathcal{D}(R)$, $\mathbb{R}\,H(C^\bullet)$ coincides with the complex $H(\underline{\mathbf{i}}C^\bullet)$, where we still denote by H its extension to $\mathcal{K}(R)$. Similarly, for any right exact covariant functor $G : \text{Mod-}S \to \text{Mod-}R$, we denote by $\mathbb{L}\,G$ its total left derived functor defined on $\mathcal{D}(S)$. For any $N^\bullet \in \mathcal{D}(S)$, $\mathbb{L}\,G(N^\bullet)$ coincides with the complex $G(\underline{\mathbf{p}}N^\bullet)$.

A module M in Mod-R is called *H-acyclic* if $R^iHM := H^i(\mathbb{R}\,HM) = 0$ for any $i \neq 0$. The abelian group R^iHM coincides with the usual i-th derived functor $H^{(i)}(-)$ of H evaluated in M. Analogously *G-acyclic* objects are defined and $L^iG(-) := H^i(\mathbb{L}\,G(-)) = G^{(-i)}(-)$. In view of these consideration, by Lemma 1.5 we have immediately the following result.

Corollary 2.1. *Let T_R be a good n-tilting module with endomorphism ring S. Then for each injective module I_R and each projective module P_S we have*

1. $\text{Hom}_R(T, I)$ *is* $- \otimes_S T$*-acyclic;*
2. $P \otimes_S T$ *is* $\text{Hom}_R(T, -)$*-acyclic.*

In particular for cochain complexes $I^\bullet$ and $P^\bullet$ whose terms are injective right R-modules and projective right S-modules respectively, we have

$$\mathbb{R}\,\text{Hom}(T, I^\bullet)\otimes^{\mathbb{L}}_S T = \text{Hom}(T, I^\bullet)\otimes_S T \text{ and } \mathbb{R}\,\text{Hom}(T, P^\bullet\otimes^{\mathbb{L}}_S T) = \text{Hom}(T, P^\bullet\otimes_S T).$$

Finally, we recall that any adjoint pair of functors (G, H) between categories of modules induces an adjoint pair $(\mathbb{L}\,G, \mathbb{R}\,H)$ between the associated unbounded derived categories. For other notations and results in derived categories we refer to [18, 23].

In the sequel we denote by H the functor $\text{Hom}_R(T, -)$ and by G the functor $- \otimes_S T$.

Theorem 2.2. *Let T_R be a good n-tilting module and $S = \text{End}\,T_R$. The following hold:*

1. *The counit adjunction morphism*
$$\mathbb{L}G \circ \mathbb{R}H \to Id_{\mathcal{D}(R)}$$
is invertible;
2. *the functor $\mathbb{R}H : \mathcal{D}(R) \to \mathcal{D}(S)$ is fully faithful;*
3. *if Σ is the system of morphisms $u \in \mathcal{D}(S)$ such that $\mathbb{L}Gu$ is invertible in $\mathcal{D}(R)$, then Σ admits a calculus of left fractions and the category $\mathcal{D}(S)[\Sigma^{-1}]$ coincides with the quotient category $\mathcal{D}(S)$ modulo the full triangulated subcategory $\text{Ker}(\mathbb{L}G)$ of the objects annihilated by the functor $\mathbb{L}G$;*

(4) *there is a triangle equivalence*

$$\mathcal{D}(S)[\Sigma^{-1}] \underset{\mathbb{R}H}{\overset{\Theta}{\rightleftarrows}} \mathcal{D}(R)$$

where Θ is the functor such that $\mathbb{L}G = \Theta \circ q$ with q the canonical quotient functor $q : \mathcal{D}(S) \to \mathcal{D}(S)[\Sigma^{-1}]$.

Proof. (1) Let $M^\bullet$ be a complex in $\mathcal{D}(R)$ and consider a K-injective resolution $\underline{\mathbf{i}}M^\bullet$ of $M^\bullet$. By Corollary 2.1 we have

$$\mathbb{L}G(\mathbb{R}H(M^\bullet)) = \mathbb{L}G(H(\underline{\mathbf{i}}M^\bullet)) = G(H(\underline{\mathbf{i}}M^\bullet)).$$

Since $(\mathrm{Hom}_R(T, I^n) \otimes_S T)_{n\in\mathbb{Z}}$ and $\underline{\mathbf{i}}M^\bullet$ are isomorphic by Lemma 1.5, (2), we have

$$\mathbb{L}G(\mathbb{R}H(M^\bullet)) = G(H(\underline{\mathbf{i}}M^\bullet)) \cong \underline{\mathbf{i}}M^\bullet = M^\bullet.$$

Conditions (2), (3) and (4) follow by applying [13, Proposition I.1.3]. $\square$

Let $\mathcal{C}$ be a triangulated category closed under arbitrary coproducts; recall that a triangle functor $L : \mathcal{C} \to \mathcal{C}$ is a *Bousfield localization* if there exists a natural transformation $\phi : 1_\mathcal{C} \to L$ such that for each X in $\mathcal{C}$

(1) $L(\phi_X) : L(X) \to L^2(X)$ is an isomorphism;
(2) $L(\phi_X) = \phi_{L(X)}$.

In such a case the kernel $\mathcal{L}$ of L is a full triangulated subcategory of $\mathcal{C}$ closed under coproducts, i.e. it is a *localizing* subcategory. The category

$$\mathcal{L}_\perp := \{X \in \mathcal{C} : \mathrm{Hom}_\mathcal{C}(\mathcal{L}, X) = 0\}$$

is called the subcategory of $\mathcal{L}$-local objects. If also $\mathcal{L}_\perp$ is closed under coproducts, then $\mathcal{L}$ is called *smashing* [6, 5].

A localization functor L factorizes as

$$\mathcal{C} \xrightarrow{q} \mathcal{C}/\operatorname{Ker} L \underset{\cong}{\xrightarrow{\rho}} \mathcal{L}_\perp \overset{j}{\hookrightarrow} \mathcal{C}$$

where q is the canonical quotient functor and ρ is an equivalence; $(\rho \circ q, j)$ is an adjoint pair. Moreover the composition

$$\mathcal{L}_\perp \overset{j}{\hookrightarrow} \mathcal{C} \xrightarrow{q} \mathcal{C}/\operatorname{Ker} L$$

is an equivalence and $(q, j \circ \rho)$ is an adjoint pair (see [5, Section 4], or [1, Proposition 1.6], or [19, Propositions 4.9.1, 4.11.1]).

Theorem 2.3. *Let (Φ, Ψ) be an adjoint pair of covariant functors between triangulated categories*

$$\mathcal{C} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{D}.$$

Denote by $\phi : 1_\mathcal{C} \to \Psi \circ \Phi$ and $\psi : \Phi \circ \Psi \to 1_\mathcal{D}$ the corresponding unit and counit. If ψ is a natural isomorphism, then the functor $L := \Psi \circ \Phi$ is a localization functor with kernel $\mathcal{L} = \operatorname{Ker} \Phi$. The functor Ψ factorizes through $\mathcal{L}_\perp$ as $\Psi = j \circ \overline{\Psi}$, where j is the inclusion $\mathcal{L}_\perp \hookrightarrow \mathcal{C}$. Finally we have a triangle equivalence

$$\mathcal{L}_\perp \underset{\overline{\Psi}}{\overset{\Phi\circ j}{\rightleftarrows}} \mathcal{D}$$

where $\Phi \circ j$ is the restriction of Φ to $\mathcal{L}_\perp$ and $\overline{\Psi}$ is the corestriction of Ψ to $\mathcal{L}_\perp$.

Proof. Since (Φ, Ψ) is an adjoint pair, we have

$$\psi_{\Phi(X)} \circ \Phi(\phi_X) = 1_{\Phi(X)};$$

applying the functor Ψ we get

$$\Psi(\psi_{\Phi(X)}) \circ L(\phi_X) = 1_{L(X)}.$$

On the other hand, again by the adjunction, we have

$$\Psi(\psi_{\Phi(X)}) \circ \phi_{\Psi\Phi(X)} = 1_{\Psi\Phi(X)}, \text{ i.e. } \Psi(\psi_{\Phi(X)}) \circ \phi_{L(X)} = 1_{L(X)}.$$

Since $\psi_{\Phi(X)}$ is an isomorphism by assumption, we have that for each X in $\mathcal{C}$

$$L(\phi_X) = \phi_{L(X)} = (\Psi(\psi_{\Phi(X)}))^{-1}$$

is an isomorphism. Hence L is a localization functor.

An object X belongs to $\mathcal{L} = \operatorname{Ker} L$ if and only if we have $0 = \Phi(0) = \Phi(\Psi\Phi(X)) \cong \Phi(X)$.

Next, since $L = \Psi \circ \Phi$ factorizes through $\mathcal{L}_\perp$ and $\Phi(\Psi(Y)) \cong Y$ for each Y in $\mathcal{D}$, also Ψ factorizes through $\mathcal{L}_\perp$. Therefore we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{L}_\perp \arrow[r, hook, "j"] \arrow[rr, bend left, "q\circ j", "\cong"'] & \mathcal{C} \arrow[r, "q"] \arrow[drr, "\Phi"'] & \mathcal{C}/\operatorname{Ker}\Phi \arrow[rr, "\rho", "\cong"'] \arrow[dr, "\Theta"] & & \mathcal{L}_\perp \arrow[r, hook, "j"] & \mathcal{C} \\
& & & \mathcal{D} \arrow[ur, "\overline{\Psi}"] \arrow[urr, "\Psi"'] & &
\end{tikzcd}$$

Finally $\Phi \circ j \circ \overline{\Psi} = \Phi \circ \Psi \cong 1_{\mathcal{D}}$, and $\overline{\Psi} \circ \Phi \circ j = \rho \circ q \circ j$, being a composition of two equivalences, is naturally isomorphic to $1_{\mathcal{L}_\perp}$. □

Applying Theorem 2.3 to our context we obtain the following result

Corollary 2.4. *Let T_R be a good n-tilting R-module and $S = \operatorname{End}(T)$. Denoted by $\mathcal{E}$ the kernel of $\mathbb{L}\,G$, and denoting by $\mathbb{R}\,H$ and $\mathbb{L}\,G$ also their restriction and corestriction, we have a triangulated equivalence*

$$\mathcal{D}(R) \underset{\mathbb{L}G}{\overset{\mathbb{R}H}{\rightleftarrows}} \mathcal{E}_\perp.$$

Embedding right R-modules and S-modules in $\mathcal{D}(R)$ and $\mathcal{D}(S)$ via the canonical functor, we obtain the following generalization of the Miyashita's results [21, Theorem 1.16]:

Corollary 2.5. *Let T_R be a good n-tilting R-module and $S = \operatorname{End}(T)$. Then for each $0 \leq i \leq n$ there is an equivalence*

$$KE_i \underset{\operatorname{Tor}_i^S(-,T)}{\overset{\operatorname{Ext}_R^i(T,-)}{\rightleftarrows}} KT_i \cap \mathcal{E}_\perp$$

Proof. Let $M \in KE_i$; then by Corollary 2.4, $\mathbb{R}\,H(M) = R^iH(M)[-i] = \operatorname{Ext}_R^i(T, M)[-i]$ belongs to $\mathcal{E}_\perp$. Since $\mathcal{E}_\perp$ is closed under shift, $\operatorname{Ext}_R^i(T, M) \in \mathcal{E}_\perp$. In $\mathcal{D}(R)$, by Theorem 2.4, (1), we have

$$M \cong \mathbb{L}\,G\,\mathbb{R}\,H(M) = \mathbb{L}\,G(\operatorname{Ext}_R^i(T, M)[-i]);$$

then for each $j \neq 0$

$$0 = H^j \, \mathbb{L}\, G(\mathrm{Ext}^i_R(T,M)[-i]) = H^{j-i} \, \mathbb{L}\, G(\mathrm{Ext}^i_R(T,M)) = \mathrm{Tor}^S_{i-j}(\mathrm{Ext}^i_R(T,M),T).$$

Therefore $\mathrm{Ext}^i_R(T,M)$ belongs to $KT_i \cap \mathcal{E}_\perp$ and $M \cong \mathrm{Tor}^S_i(\mathrm{Ext}^i_R(T,M),T)$. Analogously if $N \in KT_i \cap \mathcal{E}_\perp$, then

$$\mathbb{L}\, G(N) = L^{-i}G(N)[i] = \mathrm{Tor}^S_i(N,T)[i]$$

and since $\mathbb{R}\, H \,\mathbb{L}\, G(N) = N$ in $\mathcal{D}(S)$, necessarily $\mathrm{Tor}^S_i(N,T)$ belongs to KE_i and $N \cong \mathrm{Ext}^i_R(T, \mathrm{Tor}^S_i(N,T))$. $\square$

Proposition 2.6. *The following are equivalent:*

(1) T_R *is a classical* n*-tilting;*
(2) $\mathcal{E} = 0$ *or equivalently* $\mathcal{E}_\perp = \mathcal{D}(S)$*;*
(3) *the class* $\mathcal{E}$ *is smashing.*

Proof. $(1 \Rightarrow 2)$. Let $N^\bullet$ be a complex in $\mathcal{E}$ and $\underline{\mathbf{p}}N^\bullet$ a K-projective resolution of $N^\bullet$. By Lemma 1.5, (3) and (4), we have

$$0 = \mathbb{R}H(\mathbb{L}GN^\bullet) = \mathbb{R}H(\mathbb{L}G\underline{\mathbf{p}}N^\bullet) = \mathbb{R}H(\underline{\mathbf{p}}N^\bullet \otimes_S T) =$$

$$= \mathrm{Hom}_R(T, \underline{\mathbf{p}}N^\bullet \otimes_S T) \cong \underline{\mathbf{p}}N^\bullet = N^\bullet.$$

We conclude that $\mathcal{E} = 0$ by Corollary 2.4.

$(2 \Rightarrow 3)$ is obvious.

$(3 \Rightarrow 2)$. Since $S = \mathbb{R}\, H(T_R)$, $\mathcal{E}_\perp$ contains the bounded complexes of finitely generated projective S-modules, that is $\mathcal{E}_\perp$ containe the set $\mathcal{T}^c$ of the compact objects of $\mathcal{D}(S)$.

Since $\mathcal{D}(S)$ is compactly generated by $\mathcal{T}^c$, $\mathcal{D}(S)$ is the smallest triangulated category closed under coproducts and containig $\mathcal{T}^c$. Thus, if $\mathcal{E}_\perp$ is closed under coproducts we get that $\mathcal{E}_\perp = \mathcal{D}(S)$, hence $\mathcal{E} = 0$

$(2 \Rightarrow 1)$. By [22, Propositions 6.2, 6.3 and Theorem 6.4] for any equivalence

$$\mathcal{D}^b(R) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathcal{D}^b(S)$$

it is $\Psi = \mathbb{R}\,\mathrm{Hom}(\Phi(S), -)$ and $\Phi = - \otimes^{\mathbb{L}}_S \Psi(R)$ with $\Phi(S)$ isomorphic to a bounded complex of finitely generated projective R-modules. Since

$$\mathbb{L}\, G(S) = G(S) = S \otimes T = T_R,$$

we conclude that T_R is a classical n-tilting module. $\square$

References

[1] L. Alonso Tarrío, A. Jeremías López, and M. J. Souto Salorio. Localization in categories of complexes and unbounded resolutions. *Canad. J. Math.*, 52(2):225–247, 2000.

[2] S. Bazzoni. A characterization of n-cotilting and n-tilting modules. *J. Algebra*, 273(1):359–372, 2004.

[3] S. Bazzoni. Equivalences induced by infinitely generated tilting modules. Submitted, 2009.

[4] I. N. Bernšteĭn, I. M. Gel'fand, and V. A. Ponomarev. Coxeter functors, and Gabriel's theorem. *Uspehi Mat. Nauk*, 28(2(170)):19–33, 1973.

[5] M. Bökstedt and A. Neeman. Homotopy limits in triangulated categories. *Compositio Math.*, 86(2):209–234, 1993.

[6] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[7] S. Brenner and M. C. R. Butler. Generalizations of the Bernstein-Gel'fand-Ponomarev reflection functors. In *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, volume 832 of *Lecture Notes in Math.*, pages 103–169. Springer, Berlin, 1980.

[8] E. Cline, B. Parshall, and L. Scott. Derived categories and Morita theory. *J. Algebra*, 104(2):397–409, 1986.

[9] R. Colpi and J. Trlifaj. Tilting modules and tilting torsion theories. *J. Algebra*, 178(2):614–634, 1995.

[10] A. Facchini. A tilting module over commutative integral domains. *Comm. Algebra*, 15(11):2235–2250, 1987.

[11] A. Facchini. Divisible modules over integral domains. *Ark. Mat.*, 26(1):67–85, 1988.

[12] L. Fuchs. On divisible modules over domains. In *Abelian groups and modules (Udine, 1984)*, volume 287 of *CISM Courses and Lectures*, pages 341–356. Springer, Vienna, 1984.

[13] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[14] R. Göbel and J. Trlifaj. *Approximations and endomorphism algebras of modules*, volume 41 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, 2006.

[15] E. Gregorio and A. Tonolo. Weakly tilting bimodules. *Forum Math.*, 13(5):589–614, 2001.

[16] D. Happel. On the derived category of a finite-dimensional algebra. *Comment. Math. Helv.*, 62(3):339–389, 1987.

[17] D. Happel and C. M. Ringel. Tilted algebras. *Trans. Amer. Math. Soc.*, 274(2):399–443, 1982.

[18] R. Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.

[19] H. Krause. Localization for triangulated categories. Proceedings of "Workshop on Triangulated Categories", Leeds 2006, to appear.

[20] F. Mantese and A. Tonolo. Reflexivity in derived categories. Forum Math., to appear.

[21] Y. Miyashita. Tilting modules of finite projective dimension. *Math. Z.*, 193(1):113–146, 1986.

[22] J. Rickard. Morita theory for derived categories. *J. London Math. Soc. (2)*, 39(3):436–456, 1989.

[23] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

(Silvana Bazzoni) Dipartimento di Matematica Pura ed Applicata, Università di Padova, via Belzoni 7, I-35131 Padova - Italy

E-mail address, Silvana Bazzoni: bazzoni@math.unipd.it

(F. Mantese) Dipartimento di Informatica, Università degli Studi di Verona, strada Le Grazie 15, I-37134 Verona - Italy

E-mail address: francesca.mantese@univr.it

(Alberto Tonolo) Dipartimento di Matematica Pura ed Applicata, Università di Padova, via Belzoni 7, I-35131 Padova - Italy

E-mail address, Alberto Tonolo: tonolo@math.unipd.it